Soft Lattices

Faruk Karaaslan1, Naim Çağman2 and Serdar Enginoğlu3

Abstract

Soft set theory was introduced by Molodtsov in 1999 as a general mathematical tool for dealing with problems that contain uncertainty. In this paper, we define concept of a soft lattice, soft sublattice, complete soft lattice, modular soft lattice, distributive soft lattice, soft chain and study their related properties.

Keywords: Soft sets, soft sublattices, complete soft lattices, modular soft lattices, distributive soft lattices, soft chain.

1 Introduction

Soft set theory [31] was firstly introduced by Molodtsov in 1999 as a general mathematical tool for dealing with uncertainty. The operations of soft sets are defined by Maji et al. [30] and redefined by Cagman and Enginoğlu[6]. Recently, the properties and applications on the soft set theory have been studied increasingly [2, 9, 17, 34, 38]. The algebraic structure of soft set theory has also been studied in more detail [1, 4, 11, 18, 19, 21, 22, 23, 24, 25],

1Corresponding Author, Department of Mathematics, Gaziosmanpaşa University, 60250 Tokat, Turkey (e-mail: fkaraaslan0904@gop.edu.tr)

2Department of Mathematics, Gaziosmanpaşa University, 60250 Tokat, Turkey (e-mail: naim.cagman@gop.edu.tr)

3Department of Mathematics, Gaziosmanpaşa University, 60250 Tokat, Turkey (e-mail: serdar.enginoglu@gop.edu.tr)
and many interesting applications of soft set theory have been expanded by embedding
the ideas of fuzzy sets [4, 8, 12, 28, 35, 37].

The soft lattice structures are constructed by Nagarajan and Meenambigai [32] and Li
[27] over a soft set. In this paper, different than Li [27] and Nagarajan and Meenambigai
[32], we define soft lattices over a collection of soft sets by using Cagman and Enginoğlu’s
[6] operations of the soft sets. We also give an algebraical and a set-theorical definition of
soft lattices and we prove that algebraical and set-theorical definitions are equivalent. In
addition, we introduce complete soft lattice, soft sublattice, soft chain, distributive soft
lattice, modular soft lattice and discuss their related properties.

2 Soft set theory

In this section, for subsequent discussions, we have presented the basic definitions and
results of soft set theory which are taken from earlier studies [6, 30, 31].

Throughout this work, $U$ refers to an initial universe, $P(U)$ is the power set of $U$, $E$
is a set of parameters and $A \subseteq E$.

**Definition 2.1.** A function $f_A : E \rightarrow P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$, is called a soft
set over $U$.

The set of all soft sets over $U$ is denoted by $S(U)$.

**Definition 2.2.** Let $f_A \in S(U)$. If $f_A(x) = \emptyset$ for all $x \in E$, then $f_A$ is called an empty
soft set, denoted by $f_\emptyset$.

If $f_A(x) = U$ for all $x \in A$, then $f_A$ is called $A$-universal soft set, denoted by $f_A^\emptyset$.

If $A = E$, then the $A$-universal soft set is called universal soft set denoted by $f_E$.

**Definition 2.3.** Let $f_A, f_B \in S(U)$. Then, $f_A$ is a soft subset of $f_B$, denoted by $f_A \subseteq f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.

If $f_A$ and $f_B$ are equal, denoted by $f_A = f_B$, if and only if $f_A(x) = f_B(x)$ for all $x \in E$.

**Remark 2.4.** $f_A \subseteq f_B$ does not imply that every element of $f_A$ is an element of $f_B$. Therefore
the definition of classical subset is not valid for the soft subset. For example,
let $U = \{u_1, u_2, u_3, u_4\}$ be a universal set of objects and $E = \{x_1, x_2, x_3\}$ be the set
of all parameters. If $A = \{x_1\}$ and $B = \{x_1, x_3\}$, and $f_A = \{(x_1, \{u_2, u_4\})\}$, $f_B =
\{(x_1, \{u_2, u_3, u_4\}), (x_3, \{u_1, u_5\})\}$, then for all $e \in f_A, f_A(x) \subseteq f_B(x)$ is valid. Hence
$f_A \subseteq f_B$. It is clear that, $(x_1, f_A(x_1)) \in f_A$ but $(x_1, f_A(x_1)) \notin f_B$.

**Proposition 2.5.** If $f_A, f_B \in S(U)$, then

1. $f_A \subseteq f_E$
2. $f_\emptyset \subseteq f_A$
3. $f_A \subseteq f_A$
4. $f_A \subseteq f_B$ and $f_B \subseteq f_C \Rightarrow f_A \subseteq f_C$
Definition 2.6. Let \( f_A \in S(U) \). Then, soft complement of \( f_A \) is defined by \( f_A^c = f_{A^c} \) such that \( f_A(x) = f_A^c(x) = U \setminus f_A(x) \) for all \( x \in E \).

Definition 2.7. Let \( f_A, f_B \in S(U) \). Then, soft union of \( f_A \) and \( f_B \) is defined by \( f_A \hat{\cup} f_B = f_{A \hat{\cup} B} \) such that \( f_{A \hat{\cup} B}(x) = f_A(x) \cup f_B(x) \) for all \( x \in E \).

Soft intersection of \( f_A \) and \( f_B \) is defined by \( f_A \hat{\cap} f_B = f_{A \hat{\cap} B} \) such that \( f_{A \hat{\cap} B}(x) = f_A(x) \cap f_B(x) \) for all \( x \in E \).

Proposition 2.8. If \( f_A, f_B, f_C \in S(U) \), then
1. \( f_A \hat{\cap} f_A = f_A \)
2. \( f_A \hat{\cap} f_{\Phi} = f_A \)
3. \( f_A \hat{\cup} f_{\Phi} = f_{\Phi} \)
4. \( f_A \hat{\cup} f_{\Phi} = f_{\Phi} \)
5. \( f_A \hat{\cup} f_B = f_B \hat{\cup} f_A \)
6. \( (f_A \hat{\cup} f_B) \hat{\cup} f_C = f_A \hat{\cup} (f_B \hat{\cup} f_C) \)

Proposition 2.9. If \( f_A, f_B, f_C \in S(U) \), then
1. \( f_A \hat{\cap} f_A = f_A \)
2. \( f_A \hat{\cap} f_{\Phi} = f_{\Phi} \)
3. \( f_A \hat{\cup} f_{\Phi} = f_A \)
4. \( f_A \hat{\cup} f_{\Phi} = f_{\Phi} \)
5. \( f_A \hat{\cup} f_B = f_B \hat{\cup} f_A \)
6. \( (f_A \hat{\cap} f_B) \hat{\cap} f_C = f_A \hat{\cap} (f_B \hat{\cap} f_C) \)

Proposition 2.10. [6] If \( f_A, f_B, f_C \in S(U) \), then
1. \( f_A \hat{\cap} (f_B \hat{\cup} f_C) = (f_A \hat{\cup} f_B) \hat{\cap} (f_A \hat{\cup} f_C) \)
2. \( f_A \hat{\cap} (f_B \hat{\cup} f_C) = (f_A \hat{\cap} f_B) \hat{\cup} (f_A \hat{\cap} f_C) \)

3 Soft Lattices

In this section, the notion of soft lattices is introduced and several related properties and some characterization theorems are investigated.

Definition 3.1. Let \( \mathcal{L} \subseteq S(U) \), and \( \gamma \) and \( \lambda \) be two binary operations on \( \mathcal{L} \). If the set \( \mathcal{L} \) is equipped with two commutative and associative binary operations \( \gamma \) and \( \lambda \) which are connected by the absorption law, then algebraic structure \( (\mathcal{L}, \gamma, \lambda) \) is called a soft lattice.

Theorem 3.2. Let \( (\mathcal{L}, \gamma, \lambda) \) be a soft lattice and \( f_A, f_B \in \mathcal{L} \). Then
\[
f_A \lambda f_B = f_A \Leftrightarrow f_A \gamma f_B = f_B
\]
Proof.

\[ f_A \vee f_B = (f_A \wedge f_B) \vee f_B = f_B \vee (f_A \wedge f_B) = f_B \]

Conversely,

\[ f_A \wedge f_B = f_A \wedge (f_A \vee f_B) = f_A \]

\[ \square \]

Example 3.3. Let \( U = \{ u_1, u_2, u_3, u_4, u_5, u_6 \} \) and \( L = \{ f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}, f_{A_5} \} \subseteq S(U) \).
Assume that

\[
\begin{align*}
 f_{A_1} & = \{ (e_1, \{ u_1, u_2, u_3 \}), (e_2, \{ u_3, u_5 \}), (e_3, \{ u_4, u_6 \}) \} \\
 f_{A_2} & = \{ (e_1, \{ u_1, u_2, u_3 \}), (e_2, \{ u_3, u_5 \}) \} \\
 f_{A_3} & = \{ (e_1, \{ u_1, u_2, u_3 \}), (e_3, \{ u_4, u_6 \}) \} \\
 f_{A_4} & = \{ (e_1, \{ u_1, u_2 \}), (e_3, \{ u_4, u_6 \}) \} \\
 f_{A_5} & = \{ (e_1, \{ u_1 \}) \} \\
\end{align*}
\]

Then \( (L, \vee, \wedge) \) is a soft lattice. Tables of the operations are as follows, respectively:

\[
\begin{array}{c|cccccc}
 & f_{A_1} & f_{A_2} & f_{A_3} & f_{A_4} & f_{A_5} \\
\hline
f_{A_1} & f_{A_1} & f_{A_2} & f_{A_3} & f_{A_4} & f_{A_5} \\
f_{A_2} & f_{A_2} & f_{A_1} & f_{A_3} & f_{A_4} & f_{A_5} \\
f_{A_3} & f_{A_3} & f_{A_1} & f_{A_2} & f_{A_4} & f_{A_5} \\
f_{A_4} & f_{A_4} & f_{A_2} & f_{A_3} & f_{A_5} \\
f_{A_5} & f_{A_5} & f_{A_2} & f_{A_3} & f_{A_4} & f_{A_5} \\
\end{array}
\]

and

\[
\begin{array}{c|cccccc}
& f_{A_1} & f_{A_2} & f_{A_3} & f_{A_4} & f_{A_5} \\
\hline
f_{A_1} & f_{A_1} & f_{A_2} & f_{A_3} & f_{A_4} & f_{A_5} \\
f_{A_2} & f_{A_2} & f_{A_1} & f_{A_4} & f_{A_5} \\
f_{A_3} & f_{A_3} & f_{A_2} & f_{A_4} & f_{A_5} \\
f_{A_4} & f_{A_4} & f_{A_3} & f_{A_4} & f_{A_5} \\
f_{A_5} & f_{A_5} & f_{A_3} & f_{A_5} & f_{A_5} \\
\end{array}
\]

The Hasse Diagram of it appears in Figure 1.

Theorem 3.4. \( (L, \vee, \wedge) \) be a soft lattice and \( f_A, f_B \in L \). Then a relation \( \preceq \) that is defined by

\[ f_A \preceq f_B \iff f_A \wedge f_B = f_A \lor f_A \vee f_B = f_B \]

is an ordering relation on \( L \).

Proof. 1. \( \preceq \) is reflexive. \( f_A \preceq f_A \iff f_A \wedge f_A = f_A \).
Figure 1: A soft lattice structure

2. $\preceq$ is antisymmetric. Let be $f_A \preceq f_B$ and $f_B \preceq f_A$. Then from hypothesis,

$$f_A = f_A \land f_B = f_B \land f_A = f_B$$

3. $\preceq$ is transitive. Let be $f_A \preceq f_B$ and $f_B \preceq f_C$. Then

$$f_A \lor f_C = (f_A \lor f_B) \lor f_C = f_A \lor (f_B \lor f_C) = f_A \lor f_B = f_A$$

from hypothesis $f_A \preceq f_C$.

\begin{proof}
2. By Definition 3.1,

$$(f_A \land f_B) \lor f_A = f_A \lor (f_A \land f_B) = f_A$$

from Theorem 3.4. We get $f_A \land f_B \preceq f_A$. It can be show that $f_A \land f_B \preceq f_B$.

\end{proof}

The proof 2 can be made similarly way.
Theorem 3.6. Let \((\mathcal{L}, \gamma, \lambda)\) be a soft lattice and \(f_A, f_B, f_C, f_D \in \mathcal{L}\). Then
\[
f_A \preceq f_B \text{ and } f_C \preceq f_D \Rightarrow f_A \wedge f_C \preceq f_B \wedge f_D
\]

Proof. From hypothesis and Theorem 3.4, \(f_A \vee f_B = f_A\) and \(f_C \vee f_D = f_C\)
\[
(f_A \wedge f_C) \vee (f_B \wedge f_D) = \left[(f_A \wedge f_C) \vee f_B\right] \wedge f_D
= \left[f_A \wedge (f_C \vee f_B)\right] \wedge f_D
= \left[f_A \wedge (f_B \vee f_C)\right] \wedge f_D
= \left[(f_A \wedge f_B) \vee f_C\right] \wedge f_D
= (f_A \wedge f_B) \wedge (f_C \wedge f_D)
= f_A \wedge f_C
\]

Then, from Theorem 3.4, \(f_A \wedge f_C \preceq f_B \wedge f_D\). \(\square\)

Theorem 3.7. Let \((\mathcal{L}, \gamma, \lambda)\) be a soft lattice and \(f_A, f_B, f_C, f_D \in \mathcal{L}\). Then,
\[
f_B \preceq f_A \text{ and } f_D \preceq f_C \Rightarrow f_B \vee f_D \preceq f_A \vee f_C
\]

Proof. Proof is made similarly to Theorem 3.6. \(\square\)

Example 3.8. From Example 3.3. \(f_{A_2} \preceq f_{A_1}\) and \(f_{A_4} \preceq f_{A_3}\). Then \(f_{A_2} \cap f_{A_4} \preceq f_{A_1} \cap f_{A_3}\).

Lemma 3.9. Let \((\mathcal{L}, \gamma, \lambda)\) be a soft lattice and \(f_A, f_B \in \mathcal{L}\). Then, \(f_A \vee f_B\) and \(f_A \wedge f_B\) are the least upper and the greatest lower bound of \(f_A\) and \(f_B\), respectively.

Proof. From Theorem 3.5, \(f_A \wedge f_B\) and \(f_A \vee f_B\) are a lower bound and an upper bound of \(f_A\) and \(f_B\), respectively. Assume that, \(f_A \wedge f_B\) is not a greatest lower bound of \(f_A\) and \(f_B\). Then, \(f_C \in \mathcal{L}\) is exist, such that \(f_A \wedge f_B \preceq f_C \preceq f_A\) and \(f_A \wedge f_B \preceq f_C \preceq f_B\). Hence, by Theorem 3.6, \(f_C \wedge f_C \preceq f_A \wedge f_B\). Thus \(f_C \preceq f_A \wedge f_B\). That is \(f_C = f_A \wedge f_B\). This is a contradiction.

For \(f_A \vee f_B\) the proof can be made similarly. \(\square\)

Theorem 3.10. A soft lattice is a poset.

Proof. The proof is obviously, from Lemma 3.9. \(\square\)

Theorem 3.11. Let \(\mathcal{L} \subseteq S(U)\). Then, an algebraic structure \((\mathcal{L}, \gamma, \lambda, \preceq)\) is a soft lattice.

Proof. For all \(f_A, f_B\) and \(f_C \in \mathcal{L}\),
1. From Lemma 3.9,
\[
f_A \wedge f_B \preceq f_A \text{ and } f_A \wedge f_B \preceq f_B
\]
from Theorem 3.6
\[
f_A \wedge f_B \preceq f_B \wedge f_A
\]
Similarly,
\[
f_B \wedge f_A \preceq f_A \wedge f_B
\]
Then, \(f_A \wedge f_B = f_B \wedge f_A\). By the same way, the proof of \(f_A \vee f_B = f_B \vee f_A\) can be made.

10
2. From Theorem 3.5,
\[(f_A \triangleleft f_B) \triangleleft f_C \leq f_A \triangleleft f_B \triangleleft f_C \quad \text{and} \quad (f_A \triangleleft f_B) \triangleleft f_C \leq f_C \]
from Theorem 3.6,
\[(f_A \triangleleft f_B) \triangleleft f_C \leq f_B \triangleleft f_C \quad \text{(1)}\]
Also
\[(f_A \triangleleft f_B) \triangleleft f_C \leq f_A \triangleleft f_B \triangleleft f_A \quad \text{(2)}\]
from (1) and (2)
\[(f_A \triangleleft f_B) \triangleleft f_C \leq f_A \triangleleft f_B \triangleleft f_C \]
Similarly,
\[f_A \triangleleft (f_B \triangleleft f_C) \leq (f_A \triangleleft f_B) \triangleleft f_C \]
Then,
\[(f_A \triangleleft f_B) \triangleleft f_C = f_A \triangleleft (f_B \triangleleft f_C) \]
By the same way, the proof of \(f_A \triangleright (f_B \triangleright f_C) = (f_A \triangleright f_B) \triangleright f_C\) can be made.

3. From Theorem 3.5,
\[f_A \preceq (f_A \triangleright f_B) \quad \text{and} \quad f_A \preceq f_A, \]
and from Theorem 3.6,
\[f_A \preceq (f_A \triangleright f_B) \triangleleft f_A\]
Similarly,
\[(f_A \triangleright f_B) \triangleleft f_A \preceq f_A. \]
Then, \(f_A \triangleleft (f_A \triangleright f_B) = f_A\). By the same way, the proof of \(f_A \triangleright (f_A \triangleright f_B) = f_A\) can be made.

\[\square\]

**Note 3.12.** According to this theorem, a soft lattice \((\mathcal{L}, \triangleright, \triangleleft)\) has the same character with \((\mathcal{L}, \triangleright, \triangleleft, \preceq)\). Therefore, we shall identify any soft lattice \((\mathcal{L}, \triangleright, \triangleleft)\) with \((\mathcal{L}, \triangleright, \triangleleft, \preceq)\) and use these two concepts as interchangeable.

**Lemma 3.13.** Let \(\mathcal{L} \subseteq S(U)\). Then, soft inclusion relation \(\tilde{\subseteq}\) that is defined by
\[f_A \tilde{\subseteq} f_B \iff f_A \tilde{\cup} f_B = f_B \quad \text{or} \quad f_A \tilde{\cap} f_B = f_A\]
is an ordering relation on \(\mathcal{L}\).

**Proof.** For all \(f_A, f_B\) and \(f_C \in \mathcal{L}\),
1. \(\tilde{\subseteq}\) is reflexive. \(f_A \tilde{\subseteq} f_A\)
2. \(\tilde{\subseteq}\) is antisymmetric. \(f_A \tilde{\subseteq} f_B \quad \text{and} \quad f_B \tilde{\subseteq} f_A \iff f_A = f_B\)
3. \(\tilde{\subseteq}\) is transitive. \(f_A \tilde{\subseteq} f_B \quad \text{and} \quad f_B \tilde{\subseteq} f_C \Rightarrow f_A \tilde{\subseteq} f_C\)

11
Corollary 3.14. Let \((\mathcal{L}, \cup, \cap, \tilde{\cap})\) is a soft lattice.

Definition 3.15. Let \((\mathcal{L}, \vee, \wedge, \leq)\) be a soft lattice and \(f_A \in \mathcal{L}\).
   - If \(f_A \leq f_B\) for all \(f_B \in \mathcal{L}\), then \(f_A\) is called the minimum element of \(\mathcal{L}\).
   - If \(f_B \leq f_A\) for all \(f_B \in \mathcal{L}\), then \(f_A\) is called the maximum element of \(\mathcal{L}\).

Definition 3.16. Let \((\mathcal{L}, \vee, \wedge, \leq)\) be a soft lattice. If \(f_B \leq f_A\) or \(f_A \leq f_B\) for all \(f_A, f_B \in \mathcal{L}\), then \(\mathcal{L}\) is called a soft chain.

Example 3.17. Let \(U = \{u_1, u_2, u_3, u_4, u_5, u_6\}\). \(\mathcal{L} = \{f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}, f_{A_5}\}\) and
\[
\begin{align*}
   f_{A_1} &= \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_3, u_5\}), (e_3, \{u_4, u_6\})
   
   f_{A_2} &= \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_3, u_5\})
   
   f_{A_3} &= \{(e_1, \{u_1, u_3\}), (e_3, \{u_4, u_6\})
   
   f_{A_4} &= \{(e_1, \{u_1, u_3\})
   
   f_{A_5} &= \{(e_1, \{u_1\})
\end{align*}
\]

Although, for \(S = \{f_{A_1}, f_{A_3}, f_{A_4}, f_{A_5}\}\), \((\mathcal{S}, \cup, \cap, \tilde{\cap})\) is a soft chain, \((\mathcal{L}, \cup, \cap, \tilde{\cap})\) is not a soft chain because \(f_{A_2} \) and \(f_{A_3}\) can not comparable.

Definition 3.18. Let \((\mathcal{L}, \vee, \wedge, \leq)\) be a soft lattice. If every subsets of \(\mathcal{L}\) have both a greatest lower bound and a least upper bound, then it is called complete soft lattice.

Example 3.19. Let \(U = \{u_1, u_2, u_3\}\) and \(\mathcal{L} = \{f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}\}\) such that,
\[
\begin{align*}
   f_{A_1} &= \{(e_1, \{u_1\})
   
   f_{A_2} &= \{(e_1, \{u_1, u_2\}), (e_2, \{u_3\})
   
   f_{A_3} &= \{(e_1, \{u_1, u_2\}), (e_2, \{u_2, u_3\})
   
   f_{A_4} &= \phi
\end{align*}
\]
Then \((\mathcal{L}, \cup, \cap, \tilde{\cap})\) is a complete soft lattice. Because each finite subset of \(\mathcal{L}\) has a greatest lower bound and a least upper bound.

Definition 3.20. \((\mathcal{L}, \vee, \wedge, \leq)\) be a soft lattice and \(S \subseteq \mathcal{L}\). If \(S\) is a soft lattice with the operations of \(\mathcal{L}\), then \(S\) is called a soft sublattice of \(\mathcal{L}\).

Theorem 3.21. Let \((\mathcal{L}, \vee, \wedge, \leq)\) be a soft lattice and \(S \subseteq \mathcal{L}\). If \(f_A \wedge f_B \in S\) and \(f_A \vee f_B \in S\) for all \(f_A, f_B \in S\), then \(S\) is a soft sublattice.

Proof. It is clear from Definition 3.20. □

Corollary 3.22. Every soft chain is a soft sublattice.

Corollary 3.23. Every soft lattice is a soft sublattice of itself.

Proof. Let \(S\) be a soft chain. Since any two elements of \(S\) is comparable, \(f_A \wedge f_B \in S\) and \(f_A \vee f_B \in S\), for all \(f_A, f_B \in S\). Thus \(S\) is a soft sublattice. □

Example 3.24. \(S\), given in Example 3.17, is a soft sublattice.
**Definition 3.25.** Let \((\mathcal{L}, \vee, \wedge, \leq)\) be a soft lattice and \(f_A, f_B\) and \(f_C\) ∈ \(\mathcal{L}\). If
\[
(f_A \wedge f_B) \vee (f_A \wedge f_C) \leq f_A \vee (f_A \wedge f_C)
\]
or
\[
f_A \wedge (f_A \vee f_C) \leq (f_A \wedge f_B) \vee (f_A \wedge f_C),
\]
then \(\mathcal{L}\) is called a one-side distributive soft lattice.

**Theorem 3.26.** Every soft lattice is a one-side distributive soft lattice.

**Proof.** Let \(f_A, f_B, f_C\) ∈ \(\mathcal{L}\). From Theorem 3.2 and 3.5, we have
\[
f_A \wedge f_B \leq f_A \text{ and } f_A \wedge f_B \leq f_B \wedge f_C.
\]
Since \(f_A \wedge f_B \leq f_A\) and \(f_A \wedge f_B \leq f_B \wedge f_C\), then
\[
f_A \wedge f_B = (f_A \wedge f_B) \wedge (f_A \wedge f_B) \leq f_A \wedge (f_B \wedge f_C)
\]
and also we have \(f_A \wedge f_C \leq f_A\) and \(f_A \wedge f_C \leq f_B \wedge f_C\). Since \(f_A \wedge f_C \leq f_A\) and \(f_A \wedge f_C \leq f_B \wedge f_C\), then
\[
f_A \wedge f_C = (f_A \wedge f_C) \wedge (f_A \wedge f_C) \leq f_A \wedge (f_B \wedge f_C)
\]
From (3) and (4), we get the result,
\[
(f_A \wedge f_B) \vee (f_A \wedge f_C) \leq f_A \wedge (f_B \wedge f_C)
\]
\[\square\]

**Definition 3.27.** Let \((\mathcal{L}, \vee, \wedge, \leq)\) be a soft lattice. If \(\mathcal{L}\) satisfies the following axioms, it is called distributive soft lattice:
\[
f_A \wedge (f_B \wedge f_C) = (f_A \wedge f_B) \wedge (f_A \wedge f_C)
\]
\[
f_A \vee (f_B \wedge f_C) = (f_A \wedge f_B) \vee (f_A \wedge f_C)
\]
for all \(f_A, f_B\) and \(f_C\) ∈ \(\mathcal{L}\).

**Theorem 3.28.** \((\mathcal{L}, \cup, \cap, \subseteq)\) is a soft distributive lattice.

**Proof.** Since soft intersection is distributive over soft union operation, the proof is trivial \[\square\]

**Example 3.29.** Let \(U = \{u_1, u_2, u_3, u_4, u_5, u_6\}\) and \(\mathcal{L} = \{f_0, f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}, f_{A_5}\}\), Then, \(\mathcal{L} \subseteq S(U)\) is a soft lattice with the operations \(\cup\) and \(\cap\). Assume that,
- \(f_{A_1} = \{(e_1, \{u_1, u_2, u_3, u_4\}), (e_2, \{u_3, u_5\}), (e_3, \{u_1, u_3, u_4\})\}\)
- \(f_{A_2} = \{(e_1, \{u_1, u_2, u_4\}), (e_2, \{u_3, u_5\}), (e_3, \{u_1, u_3\})\}\)
- \(f_{A_3} = \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_3, u_5\}), (e_3, \{u_1, u_4\})\}\)
- \(f_{A_4} = \{(e_1, \{u_4\}), (e_3, \{u_1, u_3\})\}\)
- \(f_{A_5} = \{(e_1, \{u_1, u_2\}), (e_2, \{u_3, u_5\})\}\)
- \(f_0 = \emptyset\)

\((\mathcal{L}, \cup, \cap, \subseteq)\) is a soft distributive lattice. The Hasse Diagram of it appears in Figure 2.
Figure 2: A soft distributive lattice structure

**Definition 3.30.** Let \((\mathcal{L}, \vee, \wedge, \preceq)\) be a soft lattice. Then \(\mathcal{L}\) is called a soft modular lattice if it satisfies the following axiom:

\[
f_C \preceq f_A \Rightarrow f_A \wedge (f_B \vee f_C) = (f_A \wedge f_B) \vee f_C
\]

for all \(f_A, f_B\) and \(f_C \in \mathcal{L}\).

**Theorem 3.31.** A distributive soft lattice, is a soft modular lattice.

**Proof.** It is clear from Definition 3.27.

Note that, modular soft lattice may not be a distributive soft lattice.

**Proof.** Let \((\mathcal{L}, \vee, \wedge, \preceq)\) be a distributive soft lattice. Then \(f_A \wedge (f_B \vee f_C) = (f_A \wedge f_B) \vee (f_A \wedge f_C)\). Hence, from Theorem 3.4, \(f_C \preceq f_A \Rightarrow f_A \wedge (f_B \vee f_C) = (f_A \wedge f_B) \vee f_C\).

**Corollary 3.32.** \((\mathcal{L}, \cup, \cap, \subseteq)\) is a soft modular lattice.

**Example 3.33.** Let \(U = \{u_1, u_2, u_3, u_4, u_5\}\) and \(\mathcal{L} = \{f_\emptyset, f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}\}\). Then \(\mathcal{L}\) is a soft lattice with the operations \(\cup\) and \(\cap\). Assume that,

\[
\begin{align*}
    f_{A_1} &= \{(e_1, \{u_1, u_2\}), (e_2, \{u_3\}), (e_3, \{u_2, u_4\}), (e_4, \{u_5\})\} \\
    f_{A_2} &= \{(e_1, \{u_1, u_2\}), (e_3, \{u_2, u_4\})\} \\
    f_{A_3} &= \{(e_3, \{u_2, u_4\})\} \\
    f_{A_4} &= \{(e_4, \{u_1, u_5\})\} \\
    f_\emptyset &= \emptyset
\end{align*}
\]

\((\mathcal{L}, \cup, \cap, \subseteq)\) is a soft modular lattice. The Hasse Diagram of it appears in Figure 3.
Theorem 3.34. Let \((\mathcal{L}, \gamma, \lambda, \preceq)\) be a modular soft lattice. Then
\[
f_A \preceq f_B \Rightarrow f_A \preceq f_B \lor (f_A \lor f_C)
\]
for all \(f_A, f_B\) and \(f_C \in \mathcal{L}\).

Proof. The theorem is clearly from Definition 3.30. \(\square\)

Example 3.35. Assume that, \((\mathcal{L}, \hat{\cup}, \hat{\cap}, \hat{\subseteq})\) is given as a modular soft lattice. Then
\[
f_A \hat{\subseteq} f_B \Rightarrow f_A \hat{\subseteq} f_B \hat{\cap} (f_A \hat{\cup} f_C).
\]

Note that, modular soft lattice may not be a distributive soft lattice.

Example 3.36. In Example 3.33, since \(f_{A_2} \cap (f_{A_3} \cup f_{A_4}) \neq (f_{A_2} \cap f_{A_3}) \cup (f_{A_3} \cap f_{A_4})\), although \((\mathcal{L}, \hat{\cup}, \hat{\cap}, \hat{\subseteq})\) is a modular soft lattice, it is not a distributive soft lattice.

4 Conclusion

The soft set theory has been applied to many fields from theoretical to practical. In this study, we defined the concept of soft lattice as an algebraic structure and as a set-theoretic and shown that these definitions are equivalent. We then investigated several related properties and some characterization theorems.

References


REFERENCES


